Diffusion cooling in a magnetic field

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Diffusion cooling of electrons in a weakly ionized plasma in the presence of a magnetic field is studied using the balance equations of momentum transfer theory, well known in ''swarm'' or test particle analysis. It is shown that for a cylindrical, axially symmetric system, the electron temperature profile can be ''hollow'' (i.e., $T_e < T_i$) and the radial ambipolar electric field can reverse to point inwards under certain conditions, reminiscent of observations in plasmas in toroidal devices at much higher temperatures.

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I. INTRODUCTION

The electron component of a weakly ionized gas in contact with absorbing boundaries may be significantly cooled for two distinct reasons.

(a) The neutral gas component may act as a selective filter, depending upon the nature of the momentum transfer cross section, allowing higher energy electrons to diffuse to the walls, leaving the remaining bulk of electrons with a lower average energy.

(b) An ambipolar potential well may be set up, which allows only the more energetic electrons to pass to the walls, again leaving the remaining electrons with a lower mean energy.

Biondi [1] studied the latter, ambipolar ''diffusion cooling'' effect in the afterglow of a microwave discharge, while Parker [2] investigated theoretically the former, free diffusion cooling. Rhymes and Crompton found significant free diffusion cooling during experimental determination of ''swarm'' diffusion coefficients using the Cavalleri tube [3], which prompted further theoretical interest , as outlined in a recent review by the author [4]. Positrons also suffer the same effect in bounded media [5].

However, it seems that no investigation of diffusion cooling has been carried out in the presence of a magnetic field. Such a study is of interest in its own right, and indeed has been the main motivating factor behind the present paper. We start with the ''swarm'' or ''test particle'' situation, for the effect of a magnetic field has not been considered even in this relatively simple case. Moreover, there are potential advantages in exploring the connection between plasma transport theory and swarm analysis, which has made such significant advances in the last 20 years (see, e.g., Ref. [7]). This is another motivating factor behind the present paper. Having said that, it has to be remarked that the original impetus for the present study was the observation in the H1-Heliac stellarator device at the Australian National University that the electron temperature profile is ''hollow,'' i.e., the electron component of the plasma is much cooler than the ions [6], and the radial ambipolar electric field points inward. These are hints that a phenomenon like diffusion cooling may be operative. However, it is the author's opinion that it is premature to pursue the hot plasma problem in full toroidal geometry when the simpler problem of diffusion cooling in a low temperature plasma in cylindrical geometry has yet to be studied. This is the context of the present paper.

An accurate kinetic theory must be capable of at least matching the highly accurate swarm experiments (for example, drift velocities are typically measured to 1% or better [8]), which is why so much effort has been devoted to a rigorous solution of Boltzmann's equation [7]. Another approach, involving fluid equations generated from Boltzmann's equation via ''momentum transfer'' theory or some other ansatz to represent the collision terms, gives transport properties and relations typically accurate to 10% or so, and has been employed more as an adjunct to elucidate physical understanding, rather than to furnish quantitative results [9]. In plasma physics, the situation can be quite different, with accuracies of 10% more than acceptable. For these reasons, we have therefore opted for a fluid equation approach. Indeed, this seems to be the first such analysis of diffusion cooling, with or without an applied magnetic field.

In Sec. II, we set up the balance equations resulting from momentum transfer theory and discuss closure problems. In Sec. III, we apply these equations to the free diffusion cooling problem, while in Sec. IV, we consider combined ambipolar and free diffusion in cylindrical geometry with an axially applied magnetic field. Particular attention is focussed upon the direction of the ambipolar electric field. The emphasis is on phenomenology and physical understanding throughout.

II. BALANCE EQUATIONS

A. Approximations and assumptions

Momentum transfer theory has a long history of fruitful application to semiquantitative descriptions of charged particle transport processes in gases [9]. At the lowest level of approximation, it consists of assuming collisional transfer terms generated by taking moments of Boltzmann's equation for real gases to be of the same mathematical *form* as for the constant collision frequency model. Higher order approximations and an internal accuracy check can be developed. In this sense, its pedigree is quite different from many phenomenological, semiempirical fluid equations, although at first sight the mathematical structure may appear similar. Quite remarkably, momentum transfer theory has never been applied to the study of hot plasmas and Coulomb interactions,

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and neither have the rigorous techniques of swarm theory; such is the gulf that has developed between the study of swarms and plasmas.

The starting point for the present discussion is Boltzmann's equation for the charged particle phase space distribution function $f(\mathbf{r},\mathbf{c},t)$,

$$\partial_t f + \mathbf{c}\cdot\boldsymbol{\nabla} f + \mathbf{a}\cdot\partial_{\mathbf{c}} f = \Sigma J(f), \tag{1}$$

where

$$\mathbf{a} = \frac{q}{m}(\mathbf{E}+\mathbf{c}\times\mathbf{B}) \tag{2}$$

is the force per unit mass, and the right hand side represents the interaction with all other particles in the plasma. At this stage, we do not need to specify the exact nature of the collision terms $J(f)$, although we do observe that if binary collisions are assumed, then there arise singular integrals due to the long range nature of the Coulomb force which must be ''cut off'' in the usual way to account in an *ad hoc* fashion for screening effects [10]. Otherwise, we note only that in this discusion all collisions are assumed *elastic*, though this approximation can be easily relaxed. Another assumption is that the neutral component remains in undisturbed equilibrium, with zero average velocity and temperature T_0.

This paper is concerned with both free and ambipolar diffusion effects in a magnetic field. The latter seems to be somewhat problematic, going by remarks in the review of Phelps [11], contradictory textbook presentations [12,13] and an entirely different way of looking at things in the eyes of upper atmospheric physicists studying dispersion of ionized meteor trails [14]. In the present paper we simply assume that there is no net space charge, requiring

$$n_i = n_e \equiv n \tag{3}$$

for a plasma containing only one one species of singly charged ions, and that the divergence of the respective ion and electron particle fluxes are equal:

$$\boldsymbol{\nabla}\cdot\Gamma_{\mathbf{i}} = \boldsymbol{\nabla}\cdot\Gamma_{\mathbf{e}} \equiv \boldsymbol{\nabla}\cdot\Gamma. \tag{4}$$

While ''pure'' ambipolar diffusion, whereby the fluxes themselves are equal, may pertain in the absence of a magnetic field, it cannot do so when a magnetic field is present, at least for those components perpendicular to $\mathbf{B}$.

B. Balance equations

In what follows, we define the average velocity,

$$\mathbf{v} = \langle \mathbf{c} \rangle, \tag{5}$$

the mean energy

$$K = \langle \tfrac{1}{2} m c^2 \rangle, \tag{6}$$

and the energy flux

$$\mathbf{J} = \tfrac{1}{2} n m \langle c^2 \mathbf{c} \rangle. \tag{7}$$

Equation (1) is multiplied by 1, $m\mathbf{c}$, and $\frac{1}{2}mc^2$ in succession, and integrated over all velocities $\mathbf{c}$, to yield the following balance equations:

The equation of continuity is the same for both ions and electrons, viz,

$$\partial_t n + \boldsymbol{\nabla}\cdot\boldsymbol{\Gamma} = 0. \tag{8}$$

The momentum balance equations for each charged species are

$$\begin{aligned}\partial_t(nm_e\mathbf{v}_e) + \boldsymbol{\nabla}\cdot nm_e\langle\mathbf{cc}\rangle + ne(\mathbf{E}+\mathbf{v}_e\times\mathbf{B})\\ = -nm_e\nu_{m,en}\mathbf{v}_e - nm_e\nu_{m,ei}(\mathbf{v}_e-\mathbf{v}_i),\end{aligned} \tag{9}$$

$$\begin{aligned}\partial_t(nm_i\mathbf{v}_i) + \boldsymbol{\nabla}\cdot nm_i\langle\mathbf{cc}\rangle - ne(\mathbf{E}+\mathbf{v}_i\times\mathbf{B})\\ = -n\mu_{in}\nu_{m,in}\mathbf{v}_i - nm_e\nu_{m,ei}(\mathbf{v}_i-\mathbf{v}_e),\end{aligned} \tag{10}$$

where $\nu_{m,en}(\epsilon_{en})$ and $\nu_{m,in}(\epsilon_{in})$ denote momentum transfer collision frequencies for electron-neutral and ion-neutral collisions respectively, and μ_{in} is the reduced mass of an ion and a neutral. The collision frequencies are functions of the respective energies in the center of mass,

$$\epsilon_{en} = \tfrac{1}{2} m_e[\langle c^2\rangle_e + \langle c^2\rangle_0], \tag{11}$$

$$\epsilon_{in} = \tfrac{1}{2} \mu_{in}[\langle c^2\rangle_i + \langle c^2\rangle_0]. \tag{12}$$

The electron-ion momentum transfer collision frequency is, within the present approximation, given by

$$\nu_{m,ei} = n\left(\frac{2}{m_e}\right)^{1/2}\pi\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{\ln\Lambda}{\epsilon_{ie}^{3/2}}, \tag{13}$$

where

$$\epsilon_{ie} = \tfrac{1}{2} m_e[\langle c^2\rangle_i + \langle c^2\rangle_e - 2\mathbf{v}_i\cdot\mathbf{v}_e], \tag{14}$$

and Λ is the familiar screening parameter.

The *energy balance equations*, on the other hand, are given by

$$\begin{aligned}\partial_t(nK_e) + \boldsymbol{\nabla}\cdot\mathbf{J}_e + ne\mathbf{E}\cdot\mathbf{v}_e\\ = -\frac{2m_e}{m_0}n\nu_{m,en}\left[K_e - \frac{3}{2}kT_0\right] - \frac{2m_e}{m_i}n\nu_{m,ei}\\ \times\left[K_e - K_i - \frac{1}{2}(m_e-m_i)\mathbf{v}_e\cdot\mathbf{v}_i\right],\end{aligned} \tag{15}$$

$$\begin{aligned}\partial_t(nK_i) + \boldsymbol{\nabla}\cdot\mathbf{J}_i - ne\mathbf{E}\cdot\mathbf{v}_i\\ = -\frac{2\mu_{in}}{m_i+m_0}n\nu_{m,in}\left[K_i - \frac{3}{2}kT_0\right] - \frac{2m_e}{m_i}n\nu_{m,ei}\\ \times\left[K_i - K_e - \frac{1}{2}(m_i-m_e)\mathbf{v}_i\cdot\mathbf{v}_e\right].\end{aligned} \tag{16}$$

Higher order moment equations would be required to determine the energy fluxes $\mathbf{J}$, but these in turn would contain unknown moments. To obviate this closure problem, we make the ansatz

$$\mathbf{J}=(1+\gamma)K\mathbf{v}, \tag{17}$$

where γ is an empirical adjustable parameter. That it is reasonable to assume this form is clear from Eq. (7) and dimensional considerations, and the fact that only the vector $\mathbf{v}$ is available to construct other vectors. The tensor moments $\langle\mathbf{cc}\rangle$ also require some specification. For electrons (upon which we focus below) it is assumed that the velocity distribution function is very nearly isotropic, an approximation which holds very well in the absence of inelastic collisions, and hence

$$\langle\mathbf{cc}\rangle\approx\tfrac{1}{3}\langle c^2\rangle\mathbf{1}, \tag{18}$$

where $\mathbf{1}$ is the unit tensor. Finally, where convenient, we shall work with temperatures rather than energies:

$$\tfrac{3}{2}kT\equiv\tfrac{1}{2}m(\langle c^2\rangle-v^2). \tag{19}$$

We now move on to adapt the above balance equations to particular circumstances.

III. FREE DIFFUSION

A. Electrons

Here we assume that there are no space charge effects, that electron-ion collisions are negligible, and that the electrons therefore diffuse freely, i.e., we are dealing with the ''swarm'' problem [7]. The situation may be analyzed from the equations in Sec. II by setting $E=0$. To simplify matters, we drop the subscript e, since only electrons are being considered at present. Thus we have

$$\partial_t n+\nabla\cdot\mathbf{\Gamma}=0, \tag{20}$$

$$\partial_t(nm\mathbf{v})+\nabla\cdot nm\langle\mathbf{cc}\rangle+ne\mathbf{v}\times\mathbf{B}=-nm\nu_m\mathbf{v}, \tag{21}$$

$$\partial_t(n\epsilon)+\nabla\cdot\mathbf{J}=-n\nu_\epsilon(\epsilon-\tfrac{3}{2}T_0), \tag{22}$$

where

$$\nu_\epsilon=\frac{2m}{m_0}\nu_m \tag{23}$$

is the collision frequency for energy transfer. As a further simplification, we have taken $K\approx\epsilon$, a good approximation for electrons, for which $m/m_0\ll 1$.

We suppose now that the electron swarm has evolved to a stage where all average properties are independent of space and time, although the number density $n=n(\mathbf{r},t)$ is still variable. Upon eliminating $\partial_t n$ from the momentum and energy balance equations using the equation of continuity, we find

$$m(\langle\mathbf{cc}\rangle-\mathbf{vv})+ne\mathbf{v}\times\mathbf{B}=-nm\nu_m\mathbf{v} \tag{24}$$

and

$$\nabla\cdot(\mathbf{J}-n\epsilon\mathbf{v})=-n\nu_\epsilon(\epsilon-\tfrac{3}{2}kT_0). \tag{25}$$

In Eq. (24),

$$m(\langle\mathbf{cc}\rangle-\mathbf{vv})\equiv k\mathbf{T}\approx kT\mathbf{1}\approx\tfrac{2}{3}\epsilon\mathbf{1} \tag{26}$$

is the temperature tensor, which to a very good approximation is a scalar, by virtue of Eq. (18), at least for elastic collisions.

At this point, we make some general, global observations: Suppose the electrons and gas are confined in a container of volume V bounded by a surface S. Upon integrating Eq. (25) over V and applying Gauss' theorem, we obtain

$$\epsilon=\frac{3}{2}T_0-\frac{1}{N\nu_\epsilon}\int\int_S(\mathbf{J}-n\epsilon\mathbf{v})\cdot\mathbf{dS}, \tag{27}$$

where N represents the total number of electrons in V at time t. The integral represents the net transport of energy, relative to the bulk motion, to the bounding surface. Unless ν_m increases with energy faster than ϵ, this net transport of energy is positive, and then $\epsilon<\tfrac{3}{2}kT_0$, i.e., we have *diffusion cooling*. The net flux is exactly zero in the special case when $\nu_m(\epsilon)\sim\epsilon$, and then the electrons and gas are in thermal equilibrium, $\epsilon=\tfrac{3}{2}kT_0$. The parametrization (17) for energy flux follows these lines: $\gamma\gtrless 0$ depending on whether ν_m varies, respectively, less rapidly than ϵ or in direct proportion to it. Specific values of γ are given shortly.

In order to calculate the mean energy, we return to the energy balance equation, and substitute for the energy flux from Eq. (17):

$$\gamma\nabla\cdot n\mathbf{v}=-n\nu_\epsilon(\epsilon-\tfrac{3}{2}kT_0). \tag{28}$$

The momentum balance equation becomes, with approximation (26),

$$n\mathbf{v}=\frac{ne}{m\nu_m}\mathbf{v}\times\mathbf{B}-\frac{(2/3)\epsilon}{m\nu_m}\nabla n, \tag{29}$$

or, equivalently,

$$n\mathbf{v}=-\mathbf{D}\cdot\nabla n-\mathbf{D_H}\times\nabla n, \tag{30}$$

where

$$\mathbf{D}\equiv D_\parallel\hat{\mathbf{B}}\hat{\mathbf{B}}+D_\perp(\mathbf{1}-\hat{\mathbf{B}}\hat{\mathbf{B}}), \tag{31}$$

$$D_\parallel\equiv\frac{2}{3}\frac{\epsilon}{m\nu_m(\epsilon)} \tag{32}$$

is the diffusion coefficient parallel to $\mathbf{B}$, while

$$D_\perp\equiv D_\parallel\Bigg/\left(1+\frac{\Omega^2}{\nu_m^2}\right) \tag{33}$$

is the diffusion coefficient perpendicular to $\mathbf{B}$, and

$$D_H\equiv D_\perp\Omega/\nu_m \tag{34}$$

is the Hall diffusion coefficient. In these expressions,

$$\mathbf{\Omega}=e\mathbf{B}/m$$

is the electron gyrofrequency.

The divergence of the particle flux $\mathbf{\Gamma}=n\mathbf{v}$ is

$$\nabla\cdot\mathbf{\Gamma}=-\mathbf{D}:\nabla\nabla n, \tag{35}$$

and notice that the Hall term does not contribute. The diffusion equation is found by substituting Eq. (36) into the equation of continuity (8),

$$\partial_t n = \mathbf{D}:\nabla\nabla n, \tag{36}$$

while the energy balance equation (28) becomes

$$-\gamma\epsilon\mathbf{D}:\nabla\nabla n = n\nu_\epsilon(\epsilon - \tfrac{3}{2}kT_0). \tag{37}$$

For definiteness we now assume a cylindrical geometry, with **B** directed along the axis, and for simplicity, a gradient of n in the radial direction only. Thus

$$\mathbf{D}:\nabla\nabla n = D_\perp \nabla^2 n. \tag{38}$$

Next assume that $n(\mathbf{r},t)=R(\mathbf{r})T(t)$ is separable in variables, and thus we have, from the diffusion equation (37),

$$\nabla^2 n = -\Lambda^2 n \tag{39}$$

$$n(\mathbf{r},t) = R(\mathbf{r})\exp(-D_\perp t/\Lambda^2), \tag{40}$$

where $R(\mathbf{r})$ is the radial component of the solution of the diffusion equation, and Λ is a characteristic diffusion length, of the order of the radius of the cylindrical vessel. [N.B.: Equation (40) generally admits a spectrum of eigenvalues Λ, and it is implicit in what follows that we are dealing with the fundamental mode, corresponding to the largest member of this spectrum.] The values of ϵ (and hence of $D_\perp$) can be found from Eq. (38), which now takes the form

$$\frac{\gamma\epsilon D_\perp}{\nu_\epsilon \Lambda^2} + \epsilon - \frac{3}{2}kT_0 = 0, \tag{41}$$

or, equivalently,

$$\frac{2}{3}\frac{\gamma\epsilon^2}{m\nu_m\nu_\epsilon\Lambda^2(1+\Omega^2/\nu_m^2)} + \epsilon - \frac{3}{2}kT_0 = 0. \tag{42}$$

In general, the collision frequencies are energy dependent, and Eq. (43) constitutes a transcendental equation for ϵ. In the special case of constant collision frequency, however, it has the analytic solution

$$\epsilon = \frac{3}{2}kT_0/(1+\alpha), \tag{43}$$

where

$$\alpha = \tfrac{1}{2}[(1+4\kappa^2)^{1/2} - 1] \tag{44}$$

and

$$\kappa^2 \equiv \frac{\gamma kT_0}{m\nu_m\nu_\epsilon\Lambda^2(1+\Omega^2/\nu_m^2)}. \tag{45}$$

This agrees with the exact result obtained from asymptotic solution of the Boltzmann equation [4] with $B=0$ if $\gamma=\frac{2}{3}$. The corresponding diffusion coefficent is

$$D_\perp = \frac{2kT_0}{m\nu_m(1+\Omega^2/\nu_m^2)[1+(1+4\kappa^2)^{1/2}]}. \tag{46}$$

Notice that Eq. (44) is of the same mathematical form as the field-free case [4], with an effective diffusion length

$$\Lambda_{eff} = \Lambda(1+\Omega^2/\nu_m^2)^{1/2}, \tag{47}$$

i.e., the imposition of a field B acts to effectively increase Λ and thus reduce κ.

It is clear from Eq. (44) that the mean energy decreases further below $\frac{3}{2}kT_0$ the greater α is, and clearly α increases with κ. Diffusion cooling can therefore be inhibited in a swarm experiment by (a) increasing the size of the vessel, thus increasing Λ and reducing κ; (b) increasing the neutral gas pressure n_0, thus increasing ν_m and decreasing κ; and (c) increasing B/n_0 and thus also increasing Ω/n_0 and Λ_{eff}.

If $B=0$ Eq. (43) can be solved for other simple models, and agreement obtained with Boltzmann equation results [4] through an appropriate choice of the parameter γ. Thus, for example, for the constant cross section model, we choose $\gamma=\frac{1}{3}$. For the case where the cross section is proportional to speed, and $\nu_m \sim \epsilon$, we have $\gamma=0$ and there is no diffusion cooling. These values of γ may be assumed to also apply when B is non zero. Thus, by Eq. (43), there is no diffusion cooling when $\nu_m \sim \epsilon$, under any circumstances. However, the factor $1+\Omega^2/\nu_m^2$ makes analytic solution of Eq. (43) a difficult proposition in general, even for simple collision models.

In the limiting case of very strong fields, however, such that $\Omega \gg \nu_m$, a quadratic equation resembling that for the constant collision frequency case holds, regardless of the energy dependence of the cross section. The only difference is that κ is replaced by κ_∞, where

$$\kappa_\infty^2 \equiv \frac{\gamma kT_0/m}{\dfrac{2m}{m_0}\Omega^2\Lambda^2}. \tag{48}$$

That is, Eqs. (43) and (44) hold again, with $\kappa \to \kappa_\infty$, while the expression for the diffusion coefficient is

$$D_\perp = \frac{2kT_0\nu_m(\epsilon)}{m\Omega^2[1+(1+4\kappa_\infty^2)^{1/2}]}. \tag{49}$$

In this limit the diffusion cooling effect is independent of gas pressure.

B. Diffusion cooling of ions

In principle, the ion component can also experiience diffusion cooling, though for practical purposes this may be neglected. Whereas for electrons, the ratio m_e/m_0 is very small, guaranteeing that the energy transfer frequency (23) is small compared with $\nu_{m,en}$, the corresponding energy transfer frequency for ions, $[2\mu_{in}/(m_i+m_0)]\nu_{m,in}$, is comparable with $\nu_{m,in}$, making for good thermal contact between ions and the neutral gas. The parameter corresponding to κ [Eq. (46)] is thus very small, resulting in negligible diffusion cooling for ions, i.e., $T_i \approx T_0$ under all conditions.

IV. AMBIPOLAR DIFFUSION COOLING

A. Zero magnetic field

Consider first the case where $B=0$, and electron-ion collisions are negligible. The electron and ion momentum balance equations, assuming as before that ϵ and $\mathbf{v}$ are independent of position and time, and that both electron and ion temperatures are isotropic, are

$$kT_e\nabla n+ne\mathbf{E}=-nm_e\nu_{m,en}\mathbf{v}_e, \tag{50}$$

$$kT_i\nabla n-ne\mathbf{E}=-n\mu_{in}\nu_{m,in}\mathbf{v}_i. \tag{51}$$

If we take the divergence of each equation, add, and recall that the divergence of ion and electron particle fluxes are assumed to equal Eq. (4), we find

$$\nabla\cdot\mathbf{\Gamma}=-D_a\nabla^2 n, \tag{52}$$

where

$$D_a\equiv\frac{kT_e+kTi}{m_e\nu m,en+\mu_{in}\nu m,in} \tag{53}$$

$$\approx\frac{kT_e+kT_i}{\mu_{in}\nu_{m,in}} \tag{54}$$

$$=D_i(1+T_e/T_i), \tag{55}$$

and

$$D_i=\frac{kT_i}{\mu_{in}\nu_{m,in}} \tag{56}$$

is the free ion diffusion coefficient. If Eq. (53) is substituted into the equation of continuity (8), we obtain

$$\partial_t n=D_a\nabla^2 n, \tag{57}$$

the usual textbook result. Notice, however, that we did not need to assume to assume equality of the particle fluxes, only equality of their divergences.

Similarly, it can be shown from the momentum balance equations that

$$e\nabla\cdot n\mathbf{E}=-\left[\frac{\dfrac{kT_e}{m_e\nu_{m,en}}-\dfrac{kT_i}{\mu_{in}\nu_{m,in}}}{\dfrac{1}{m_e\nu_{m,en}}+\dfrac{1}{\mu_{in}\nu_{i,in}}}\right]\nabla^2 n \tag{58}$$

$$\approx -kT_e\nabla^2 n. \tag{59}$$

A sufficient (but not necessary) condition for this to hold is

$$ne\mathbf{E}=-kT_e\nabla n. \tag{60}$$

We now turn to the energy balance equation for electrons. Thus Eq. (25) becomes

$$\gamma\epsilon\nabla\cdot\mathbf{\Gamma}+ne\mathbf{v}\cdot\mathbf{E}=-n\nu_\epsilon\left(\epsilon-\frac{3}{2}kT_0\right), \tag{61}$$

and upon substitution for the ambipolar field $\mathbf{E}$ and $\nabla\cdot n\mathbf{v}$, this becomes

$$-(\gamma+\tfrac{2}{3})\epsilon D_a\nabla^2 n=-n\nu_\epsilon(\epsilon-\tfrac{3}{2}kT_0). \tag{62}$$

This has the same structure as the energy balance equation (38) for the free diffusion cooling case, although D_a has an energy dependence different from the free diffussion coefficient. Moreover, even if $\gamma=0$, as is the case when $\nu_m(\epsilon)\sim\epsilon$, there is still diffusion cooling. This is because the ambipolar space-charge barrier cools no matter what the nature of the collisions. Equation (63) is to be compared with the empirical equation (9) of Biondi [1]. As before, ions suffer negligible diffusion cooling because of the good thermal contact with the neutral gas.

B. Nonzero magnetic field

In the presence of a magnetic field, ''pure'' ambipolar diffusion, in the sense that *all* components of ion and electron fluxes are equal, is not possible [12]. However, their *divergences* must balance as in Eq. (4) in order to maintain space-charge neutrality. Furthermore, since motion along $\mathbf{B}$ is unaffected by the magnetic field, we can assume that strict ambipolarity is maintained in that direction, i.e.,

$$\hat{\mathbf{B}}\cdot\mathbf{\Gamma}_i=\hat{\mathbf{B}}\cdot\mathbf{\Gamma}_e\equiv\Gamma_\parallel. \tag{63}$$

The momentum balance equations are written as

$$kT_e\nabla n+e(n\mathbf{E}+\mathbf{\Gamma_e}\times\mathbf{B})=-m_e\nu_{m,en}\mathbf{\Gamma_e}, \tag{64}$$

$$kT_i\nabla n-e(n\mathbf{E}+\mathbf{\Gamma_i}\times\mathbf{B})=-\mu_i\nu_{m,in}\mathbf{\Gamma_i}. \tag{65}$$

Taking the dot product of each of these with $\hat{\mathbf{B}}$, and applying ambipolarity along the magnetic field (64), gives the $B=0$ result, viz.,

$$\Gamma_\parallel=-D_{a,\parallel}\nabla_\parallel n, \tag{66}$$

where

$$D_{a,\parallel}=D_i(1+T_e/T_i) \tag{67}$$

and

$$E_\parallel=-\frac{kT_e}{ne}\nabla_\parallel n. \tag{68}$$

For the transverse direction, no such ambipolarity generally exists, and it is shown in the Appendix that for the axisymmetric case the equation of continuity is

$$\frac{\partial n}{\partial t}=-\nabla\cdot\mathbf{\Gamma}=-D_{a,\perp}\nabla_\perp^2 n-D_{a,\parallel}\nabla_\parallel^2 n, \tag{69}$$

where

$$D_{a,\perp}=\frac{D_{a,\parallel}}{1+\rho} \tag{70}$$

and

$$\rho \equiv \frac{e^2B^2}{m_e \nu_{m,en} \mu_i \nu_{m,in}}. \tag{71}$$

In the axisymmetric case, radial (but not azimuthal fluxes) are indeed ambipolar; that is,

$$\Gamma_{e,r} = \Gamma i,r = -D_{a,\perp} \frac{\partial n}{\partial r}, \tag{72}$$

and in that case the transverse (radial) field is also given by

$$neE_r = -\zeta k T_e \frac{\partial n}{\partial r}, \tag{73}$$

where

$$\zeta \equiv \frac{1 - \dfrac{\rho T_i}{T_e}}{1+\rho}. \tag{74}$$

The electron energy balance equation is of the same mathematical form as for the zero field case, and the effect of B becomes apparent only when explicit expressions are substituted for the flux and electric field. Thus we find, assuming uniformity along the axial direction,

$$-(\gamma + \tfrac{2}{3}\zeta)\epsilon D_{a,\perp} \nabla_\perp^2 n = -n \nu_\epsilon (\epsilon - kT_0). \tag{75}$$

Apart from the terms ζ and $D_{a,\perp}$ this is of a form similar to the field-free case [Eq. (63)]. It is interesting to note that ζ can be *negative* for sufficiently large B, which raises the interesting possibility of diffusion *heating* resulting from application of a magnetic field.

The ions can still be expected to remain in thermal equilibrium with the neutrals, i.e., $T_i \approx T_0 > T_e$, given that the collision frequency for energy exchange of ions with neutrals is several orders of magnitude larger than the corresponding quantity for electrons. Notice that for sufficiently large B, E_r can actually change sign, i.e., the ambipolar field can reverse direction from radially outwards to radially inwards. Indeed, if $\rho \gg 1$, Eqs. (74) and (75) indicate that

$$neE_r \approx kT_i \frac{\partial n}{\partial r}. \tag{76}$$

In terms of the electrostatic potential ϕ defined by $E_r = -\partial\phi/\partial r$, this then implies

$$n(r) \sim \exp(-e\phi(r)/kT_i). \tag{77}$$

That is, the ions are in thermal equilibrium with a Maxwell-Boltzmann distribution.

V. CONCLUDING REMARKS

In this paper we have given a semiquantitative analysis of a weakly ionized plasma undergoing diffusion in a finite cavity, both with and without an applied magnetic field, and for both free and ambipolar diffusion regimes. It was shown that the energy flux plays a significant role in the theory, but at this level of closure of the equations, it had to be approximated in order to make progress. For simplicity, we have taken cylindrical geometry, with **B** directed along the axis.

The main results are as follows. (a) The temperature of the electron component of the plasma can be significantly lowered through the phenomenon of ''diffusion cooling.'' (b) The magnetic field may act to inhibit diffusion cooling in the free diffusion case. (c) For the ambipolar case, the radial ambipolar electric field can actually reverse sign when B becomes large enough. (d) In all cases, the ions remain in approximate thermal equilibrium with the neutral component.

Given the essentially phenomenological nature of this initial investigation, we prefer to leave any numerical calculations for explicit situations to subsequent papers, where it is planned to develop a full kinetic theory treatment of diffusion cooling effects in a magnetic field. For the free diffusion, ''swarm'' problem, it can be shown that the results presented in Ref. [4] can be generalized in a straightforward way, by simply using an effective diffusion length as in Eq. (48). However, for the ambipolar situation, simultaneous solution of *both* electron and ion Boltzmann equations is required, and this is a matter of current investigation even in the much simpler case for which $B=0$. To add to that, inelastic collisions need to be included in any serious investigation of plasma phenomena, and again it is felt to be better to leave that to a more comprehensive investigation via numerical solution of Boltzmann equation.

Finally, to try to make contact with the hot toroidal plasma problem, the initial motivating force behind the current investigation, requires yet another step up in sophistication. All that is being pointed out here is that the diffusion cooling phenomenon has certain qualititative similarities with what has been observed in experiment [6] (''hollow'' electron temperature profile, radial ambipolar field reversing direction) without trying to claim in any way that it is *the* explanation.

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APPENDIX: AXISYMMETRIC CASE

For the sake of completeness, and because the picture presented in even standard textbooks is by no means clear or consistent, we give a brief outline of the derivation of Eqs. (72) and (73) from first principles.

Consider a cylindrical plasma, with an axial magnetic field **B** which defines the z axis of a system of cylindrical coordinates (r,θ,z). We have in mind an axially symmetric case, where the density is independent of both the azimuthal angle θ and the longitudinal coordinate z, so that that $n = n(r)$. In addition, the azimuthal component of the ambipolar field $E_\theta = 0$.

The equations of motion for electrons and ions, resolved into radial and azimuthal components are

$$kT_e \frac{\partial n}{\partial r} + e(nE_r + \Gamma_{e,\theta}B) = -m_e \nu_{m,en} \Gamma_{e,r} \tag{A1}$$

$$-e\Gamma_{e,r}B=-m_e\nu_{m,en}\Gamma_{e,\theta}, \tag{A2}$$

$$kT_i\frac{\partial n}{\partial r}-e(nE_r+\Gamma_{i,\theta}B)=-\mu_i\nu_{m,in}\Gamma_{i,r} \tag{A3}$$

$$e\Gamma_{i,r}B=-\mu_i\nu_{m,en}\Gamma_{i,\theta}. \tag{A4}$$

Eliminating $\Gamma_{e,\theta}$ and $\Gamma_{i,\theta}$ in these equations then gives

$$kT_e\frac{\partial n}{\partial r}+enE_r=-\left(m_e\nu_{m,en}+\frac{e^2B^2}{m_e\nu_{m,en}}\right)\Gamma_{e,r}, \tag{A5}$$

$$kT_i\frac{\partial n}{\partial r}-enE_r=-\left(\mu_i\nu_{m,in}+\frac{e^2B^2}{\mu_i\nu_{m,in}}\right)\Gamma_{i,r}. \tag{A6}$$

We now assume that ambipolarity pertains in the radial direction, i.e.,

$$\Gamma_{i,r}=\Gamma_{e,r}\equiv\Gamma_r. \tag{A7}$$

[Note, however, that although $\Gamma_{i,\theta}\neq\Gamma_{e,\theta}$, the equality of the divergences of the particle fluxes, Eq. (4), still holds.] Adding Eqs. (A5) and (A6) and neglecting terms of order m_e/m_i gives, for the particle flux,

$$\Gamma_r=-\frac{kT_i+kT_e}{\mu_i\nu_{m,in}+\dfrac{e^2B^2}{m_e\nu_{m,en}}}\frac{\partial n}{\partial r}, \tag{A8}$$

which is effectively Eq. (72). Subtracting Eqs. (A5) and (A6) similarly leads to Eq. (73) for the ambipolar electric field.

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